Equilibrium Distortion with Dual Noise: The Sampling Logit Approach

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Minoru Osawa

Institute of Economic Research, Kyoto University

Background and Motivation

- Models of **bounded rationality** are widely used in both theoretical and quantitative economic analysis.
  - Random Utility Models (RUM) are core tools in transport demand analysis as well as spatial economics.
- In boundedly rational choice, there are essentially **two sources of noise**:
  1. **Idiosyncratic errors**: $\approx$ Standard RUM
  2. **Limited observation**: Systematic distortion from imperfect information

How are equilibria in large games are distorted under both sources of noise?

- Environment: Large-population games (Sandholm, 2010)

More Backgrounds

1. **Idiosyncratic noise** / RUM
   $\approx$ A quantitative tool to address “irrationality” in choice data
   - McFadden in 1980s: RUM Foundation
   - Logit eqm. in routing games, aka Stochastic User Eqm. (Sheffi, 1984)
   - “Quantitative” spatial models (Redding & Rossi-Hansberg, 2017)
   - Quantal response eqm. (McKelvey & Palfrey, 1995; Goeree et al., 2005)

2. **Sampling (finite observation) noise** $\approx$ A model of micro behavior
   - Related works: Choice and equilibrium under imperfect state observation (Osborne & Rubinstein 2003; Salant & Cherry 2020), and corresponding dynamics (Oyama et al. 2015; Sawa & Wu 2023). Equilibrium selection (Kreindler and Young, 2013).

Objective

This study introduces:

- a choice rule (Sampling Logit Choice) that combines two noise sources,
- the corresponding stationary concept (Sampling Logit Equilibrium), and
- the corresponding evolutionary dynamic.

Results:

1. Suggest natural connections to **equilibrium selection** (Oyama et al. 2015).
2. Show that **“virtual” preference for variance** emerges endogenously because of noise in sampling.
3. Give **comparative statics** on how SLE depends on noise parameters.

Environment

- **Large-population game** (single population)
  - Homogeneous and anonymous continuum agents, and each chooses a pure action $i \in S \equiv \{1, 2, \dots, n\}$
  - **Population state** is a distribution $x \in X = \{x \geq 0 \mid \sum_i x_i = 1\}$.
  - **Payoff function** $x \mapsto F(x) = (F_i(x))_{i=1}^n$
    - Assumption: All convenient properties
- Given S, the payoff function F fully identifies the game.
- Fits well in the context of cities and transport.

Examples

- Random matching in symmetric normal-form games $A = [a_{ij}]$
  - Expected payoff
$$F_i(x) = \sum_j a_{ij} x_j \quad \text{or} \quad F(x) = Ax$$
  - A identifies the game
- Congestion games
  - "Network equilibrium" in transport engineering (Beckmann et al., 1956)
  - For example, S is the set of alternative routes over network
  - The payoff of route $i \in S$:
$$F_i(x) = -\mathrm{TravelCost}_i(x)$$

Nash Equilibrium and Sampling Equilibrium

- **Nash Equilibrium (NE):** $x \in \mathrm{BR}(x)$
  - BR is the mixed-strategy best response:

$$\mathrm{BR}(x) = \{y \in X : y_i > 0 \Rightarrow i \in \arg\max_k F_k(x)\}.$$

- **k-Sampling Equilibrium (SE):** $x \in \mathrm{BR}^k(x)$

1. Each agent observes k others:
   Counts distribution $z \sim \mathrm{Multinomial}(k, x)$.
2. Forms the ML estimate, i.e., empirical distribution $w = \frac{1}{k} z$.
3. Best responds to inferred payoffs $F(w)$:

$$\mathrm{BR}^k(x) = \mathbb{E}[\mathrm{BR}(w)] = \sum_z \Pr(z)\,\mathrm{BR}(w).$$

Logit Equilibrium and Sampling Logit Equilibrium

- **η-Logit Equilibrium (LE):** $x = P^\eta(x)$

$$P_i^\eta(x) = \frac{\exp(\eta^{-1} F_i(x))}{\sum_l \exp(\eta^{-1} F_l(x))}.$$

- **(k, η)-Sampling Logit Equilibrium (SLE):** $x = L^{k,\eta}(x)$

1. Each agent observes k others:
   Counts distribution $z \sim \mathrm{Multinomial}(k, x)$.
2. Forms the ML estimate, i.e., empirical distribution $w = \frac{1}{k} z$.
3. **Logit responds** to inferred payoffs $F(w)$:

$$L^{k,\eta}(x) = \mathbb{E}[P^\eta(w)] = \sum_z \Pr(z) P^\eta(w)$$

Corresponding Myopic Dynamics

- **Best Response (BR) Dynamic** (Gilboa and Matsui, 1991; Hofbauer, 1995)

$$\dot{x} \in \mathrm{BR}(x) - x$$

- **k-Sampling BR Dynamic** (Oyama, Sandholm, and Tercieux, 2015)

$$\dot{x} \in \mathrm{BR}^k(x) - x$$

- **Logit Dynamic** (Fudenberg and Levine, 1998, Ch.4)

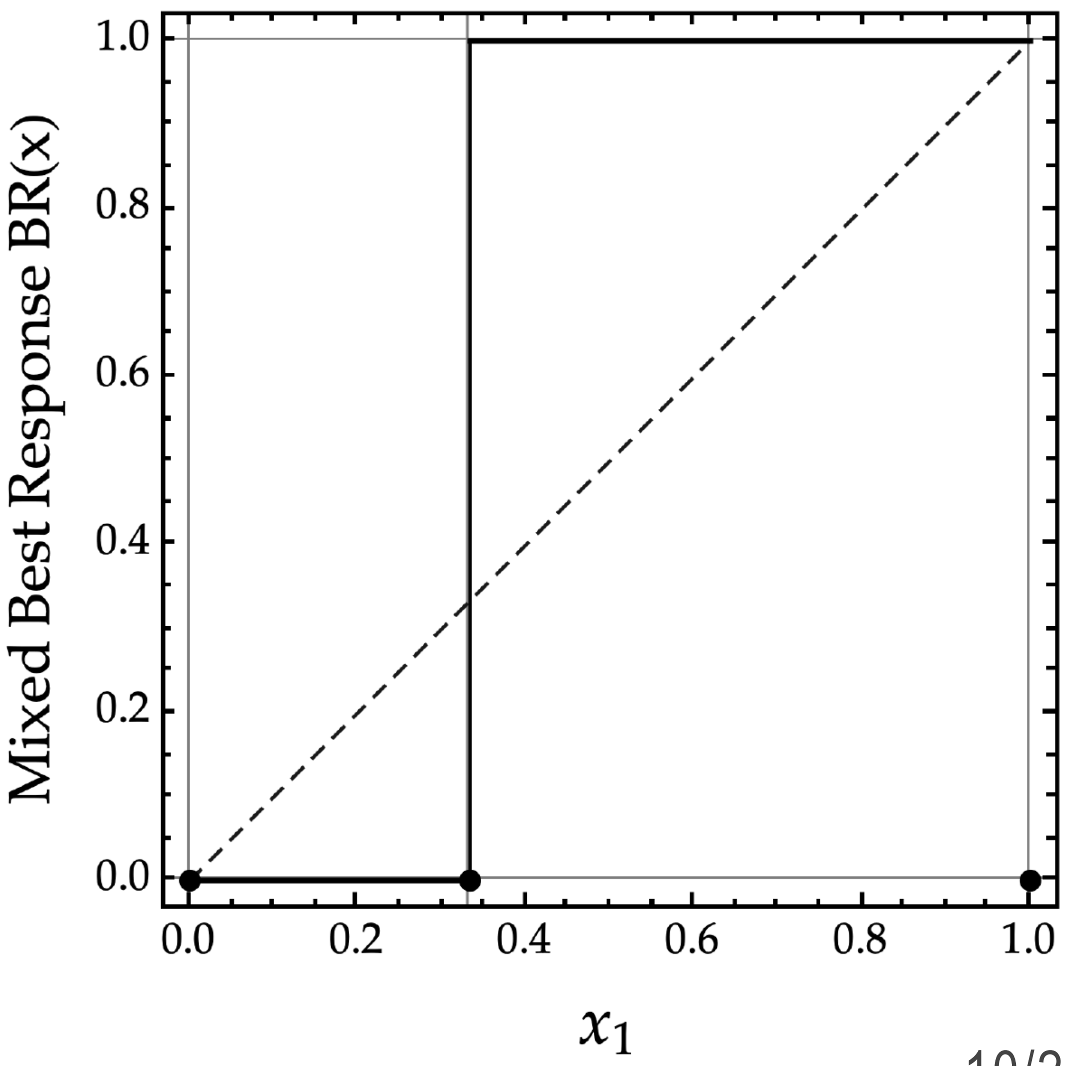
$$\dot{x} = P^\eta(x) - x$$

- **Sampling Logit Dynamic** (This study)

$$\dot{x} = L^{k,\eta}(x) - x$$

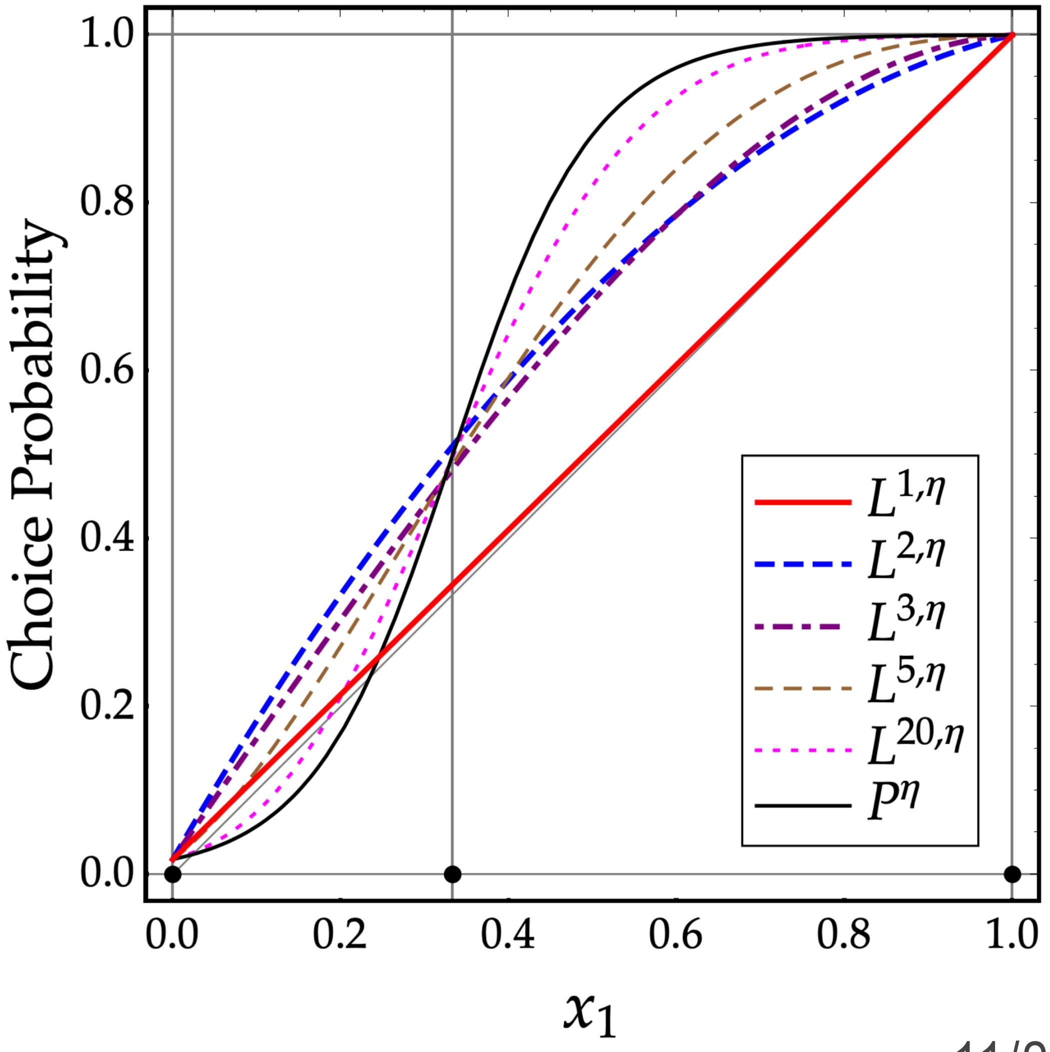
Example 1: A Simple 2×2 Coordination Game

- Suppose $F(x) = Ax$ with $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$
  - Or $F_1(x) = 2x_1$ and $F_2(x) = x_2$.
- Nash Eqms: $x_1 \in \{0, 1/3, 1\}$.
- Under the Best Response Dynamic,
  - $x_1 = 1/3$ is locally unstable
  - $x_1 \in \{0, 1\}$ are locally stable
- $x_1 = 1$ is **risk dominant**
  - Selected under various rules



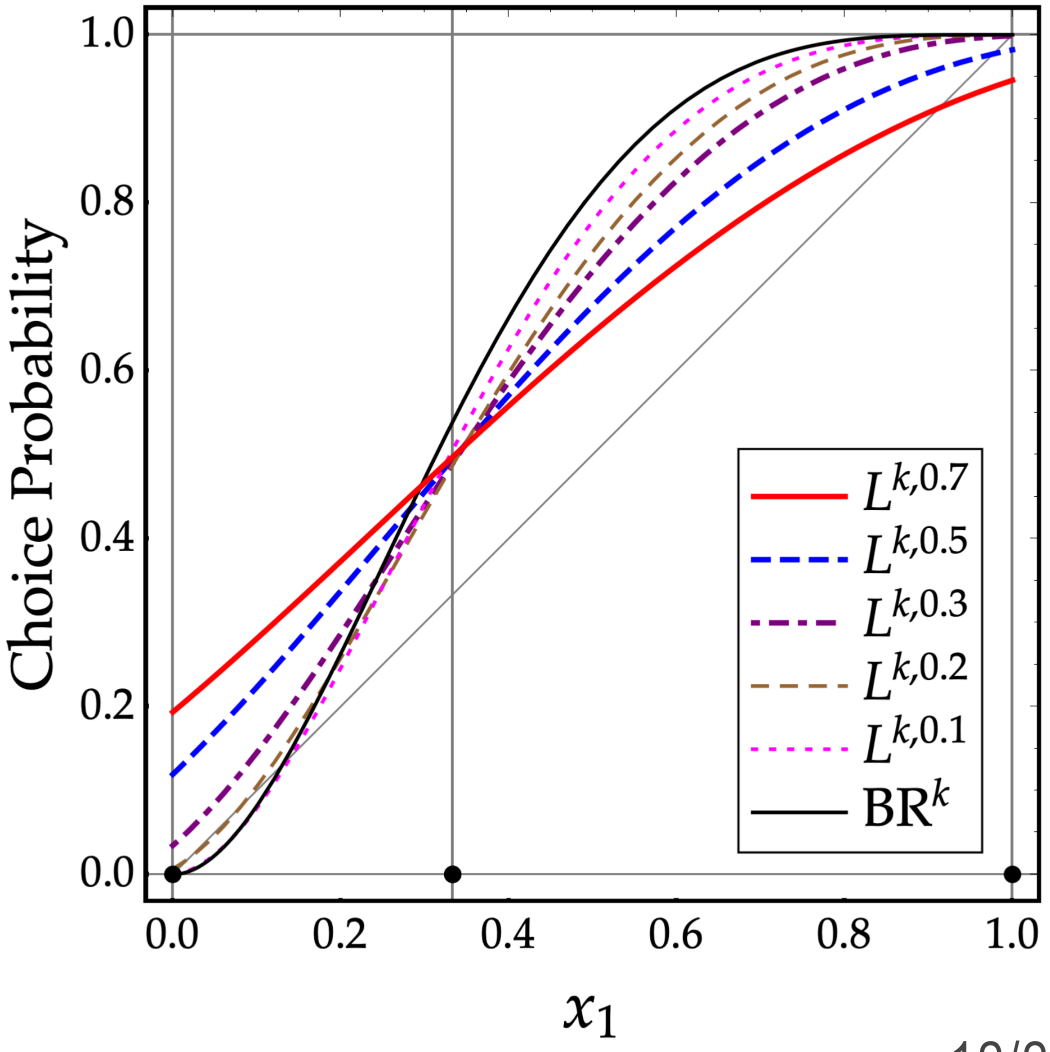
Example 1: Choice Probability for Action 1

- s-logit vs. logit at $\eta = 0.25$
- $L_1^{k,0.25}(x) \to P_1^{0.25}(x)$ as $k \to \infty$



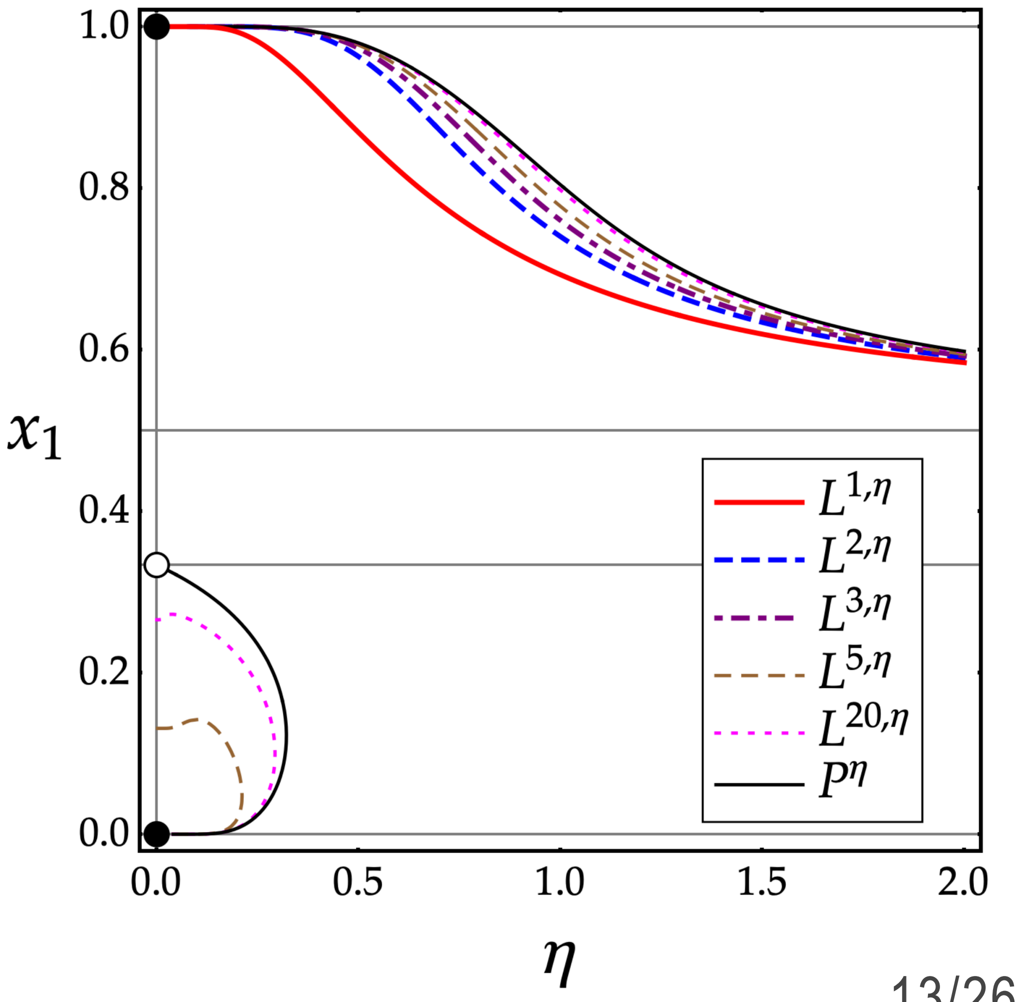
Example 1: Choice Probability for Action 1

- s-logit vs. s-BR at $k = 5$
- $L_1^{5,\eta}(x) \to \mathrm{BR}_1^{5}(x)$ as $\eta \to 0$



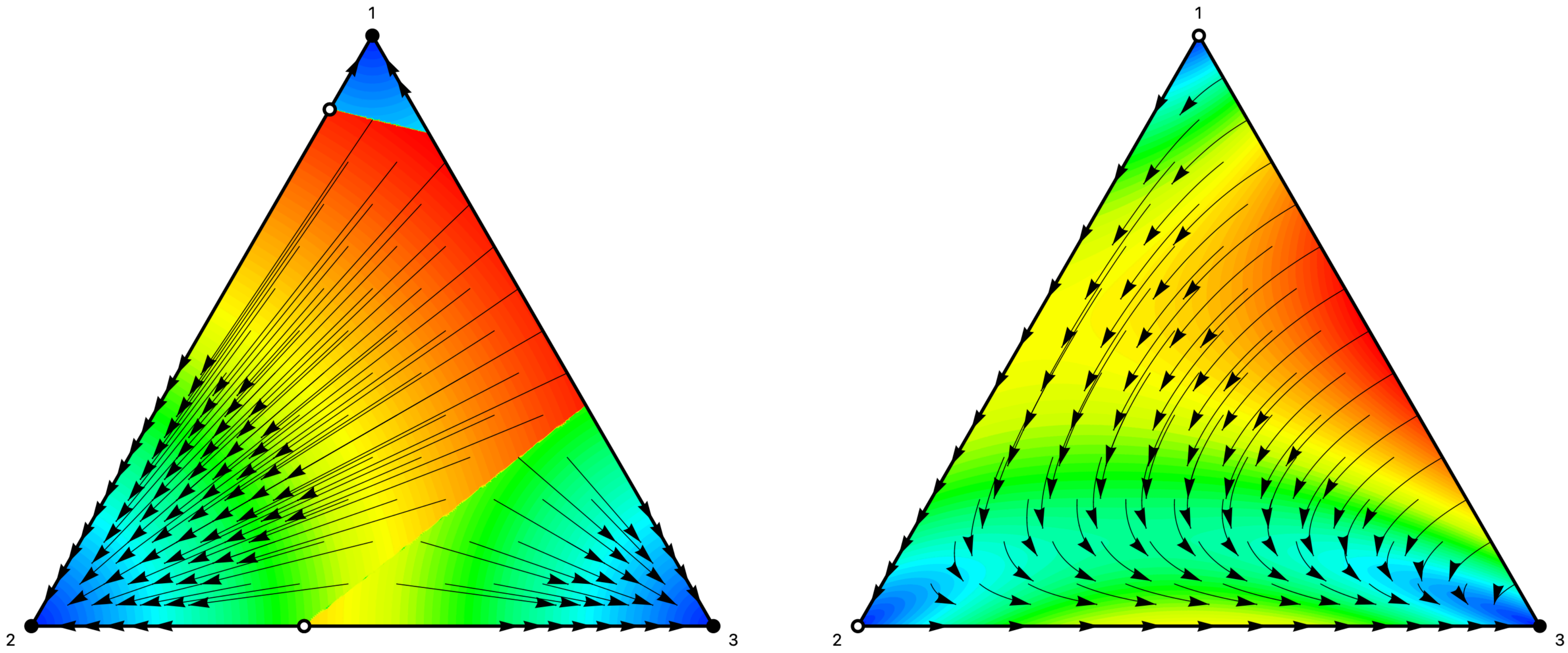
Example 1: Equilibria and Selection

- Fact: LE $\to$ NE as $\eta \to 0$ / SE $\to$ NE as $k \to \infty$
- Natural properties of SLE
  - $\to$ LE as $k \to \infty$
  - $\to$ SE as $\eta \to 0$
  - $\to$ approx NE as $\eta \to 0$ (if k is large)
- Limiting SLE yields **equilibrium selection** as η goes down from relatively high level, provided that k is relatively small.
  - cf. Kreindler and Young (2013, Sec. 6)



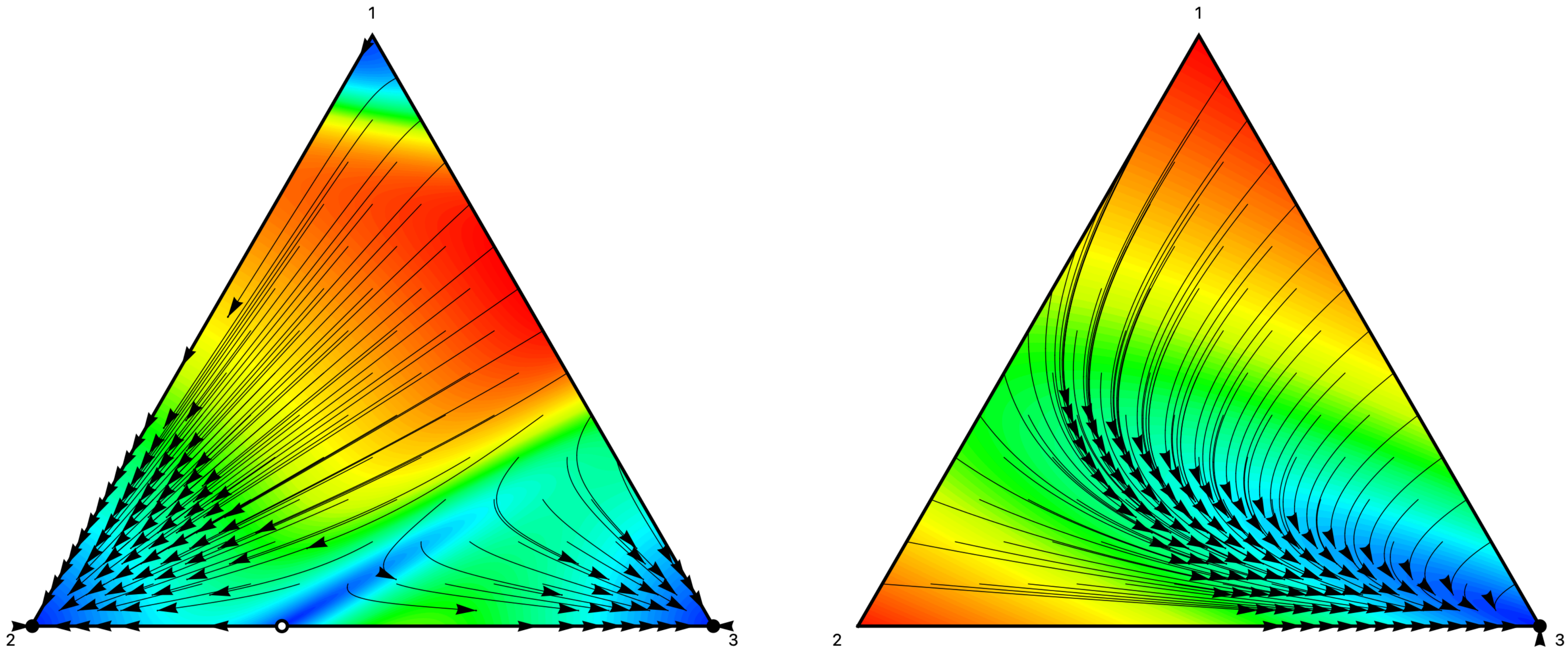
Example 2: Young (1993)'s 3×3 Game

- The BR dynamic and the sampling BR dynamic
- “Almost global” stability of $x = e_3 = (0, 0, 1)$ for small k (Oyama et al., 2015)



Example: Young (1993)'s 3×3 Game (2/2)

- The logit dynamic and the sampling logit dynamic
- Global stability of $x = e_3$ for small k. Maybe faster? (No formal analysis yet)



Summary so far

- Sampling logit chioce: A natural extension of sampling best response rule.
- The associated equilibrium concepts follow naturally.
  - Nash Eqm. $\rightarrow_k$ Sampling Eqm.
    $\downarrow_\eta$ $\downarrow_\eta$
    Logit Eqm. $\rightarrow_k$ Sampling Logit Eqm.
- Natural analogues to known results on equilibrium selection:
  - Selection of the risk-dominant (RD) eqm. in logit QRE (Turocy, 1995)
  - Selection of $(1/k)$-dominant eqm. under sampling BR (Oyama et al. 2015)
  - Fast convergence (Kreindler and Young, 2013, Sec. 6)
- But how and why the two kinds of noise distort equilibrium?

How to Understand $L^{k,\eta}$?

- Need to understand the choice rule $L^{k,\eta}$:

$$L^{k,\eta}(x) = \mathbb{E}[P^{\eta}(w)], \quad w = \frac{1}{k} z.$$

- For large k, we can approximate $w \sim \mathrm{Normal}(x, \frac{1}{k}\Sigma(x))$.
  - $\mathbb{E}[w] = x$ and $\mathrm{Var}[w] = \frac{1}{k}\Sigma$, where $\Sigma(x) = \mathrm{Var}[z] = \mathrm{diag}[x] - xx^{\top}$
- Then, by **the delta method** (e.g., van der Vaart, 2000, Ch.3), we can approximate:

$$L^{k,\eta}(x) \approx \widetilde{L}(x) = \mathbb{E}\left[\text{Taylor approximation of } P^{\eta}(w) \text{ about } x\right].$$

- $\widetilde{L}(x)$ is a function of x, $\Sigma(x)$, and info of F at x. And of course (k, η)
  = A relatively simple function of x and (k, η) !
  - Could be easier to understand.

Simplification and Notations

- For simplicity, we focus on the linear case $F(x) = Ax$.
  - $\widetilde{L}$ is relatively simple for this case. The nonlinear case is in the paper.
- Some notations. For any collection $\{y_i\}_{i=1}^n$, set
  - Logit weighted mean at x :

$$\overline{y}(x) = \sum_{i=1}^{n} P_i^{\eta}(x) y_i$$

  - Relative values at x :

$$\widehat{y_i}(x) = y_i - \overline{y}(x)$$

Approximation Formula via the Delta Method

- **Theorem 1**: For k sufficiently large,

$$L^{k,\eta}(x) \approx \widetilde{L}_i(x) = \left(1 + \frac{1}{2\eta^2}\,\widehat{\sigma}_i(x)\right) P_i^{\eta}(x).$$

- The η-logit choice rule P^{η} with a multiplicative correction term.
- $\sigma_i(x) > 0$ is the **variance of relative marginal payoffs** at x:

$$\sigma_i(x) = \frac{1}{k} \cdot \widehat{A_i}(x)^{\top} \Sigma(x) \widehat{A_i}(x)$$

  - For $F(x) = Ax$, we have $\nabla F_i(x) = A_i = (a_{il})_{l=1}^{n}$.

- Reduces to P^{η} when $k \to \infty$. Also, $k\eta \to \infty$ required for accuracy.

Variance Premium

- Variance premium: Actions with higher relative marginal payoff variances are chosen more often than the plain η-logit choice rule P^η.

$$L^{k,\eta}(x) \approx \widetilde{L}_i(x) = \left(1 + \frac{1}{2\eta^2}\,\widehat{\sigma}_i(x)\right) P_i^\eta(x).$$

- **On average,** agents exhibit bias toward “risky” options.
  - Not an individual-level behavior, but a population-level effect.
  - Aggregate “preference” for variance arises endogenously.

Virtual Payoff Representation

- The induced bias can be written as a “virtual” payoff primitive.
  - cf. Hofbauer and Sandholm (2007, Appendix): “Virtual payoffs” = An equivalent $\log$ penalty representation of logit equilibria.
- Set $\widetilde{L}_i(x) = \left(1 + \frac{1}{2\eta^2}\widehat{\sigma}_i(x)\right) P_i^{\eta}(x) = G_i(x) P_i^{\eta}(x)$.
- Further, set $\widetilde{F}_i(x) \equiv F_i(x) + \eta \log G_i(x)$. Then,

$$\widetilde{L}(x) = \frac{\exp(\eta^{-1}\widetilde{F}_i(x))}{\sum_l \exp(\eta^{-1}\widetilde{F}_l(x))}.$$

- Fixed point $x = \widetilde{L}(x) \Leftrightarrow \eta$-logit equilibrium of the virtual game $\widetilde{F}$!

Why Does Variance Premium Emerge?

- Consider $p(\mu) \equiv \exp(\eta^{-1}\mu)$, where μ is some payoff.
- Consider estimation error: $\mu + \epsilon$, where $\epsilon = \pm\zeta$ with equal prob.
- Positive errors increase p more than negative errors decrease it:

$$p(\mu + \zeta) - p(\mu) \geq p(\mu) - p(\mu - \zeta).$$

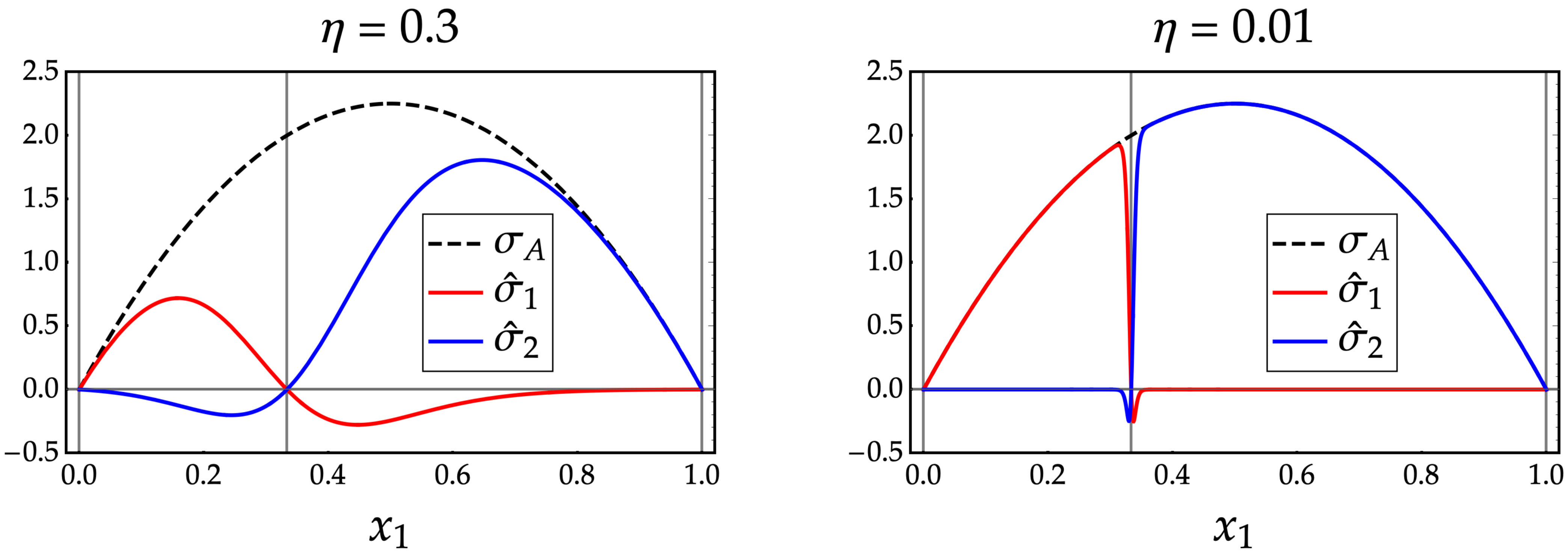
- Expected value $\mathbb{E}[p(\mu + \epsilon)]$ is **upward-biased** (basically Jensen's ineq.):

$$\mathbb{E}[p(\mu + \epsilon)] \geq p(\mu)$$

- Also, we can show this bias is $\propto \mathrm{Var}[\epsilon]$.
- Futher, $\mathrm{Var}[\epsilon] = \mathrm{Var}[F_i(w) - F_i(x)] \approx \mathrm{Var}[\nabla F_i(x)(w - x)] \Rightarrow \sigma_i(x)$
- P^μ are relative values of $p(\eta^{-1}F_i(x)) \rightarrow$ relative values ($\widehat{\ }$) matter.

Example 1 (Cont'd): 2×2 Coordination Game

- $\widetilde{L}_1(x) = (1 + c\,\widehat{\sigma_1}(x))P_1(x)$ and $\widetilde{L}_2(x) = (1 + c\,\widehat{\sigma_2}(x))P_2(x)$



- Ex.: $\mathrm{BR}(x) = \{2\}$ for $x_1 < 1/3$. However, $\widehat{\sigma_1}(x) > 0$ (shifts P_1^{η} upwords).
- Payoff estimation errors introduce **bias toward the suboptimal choice**.

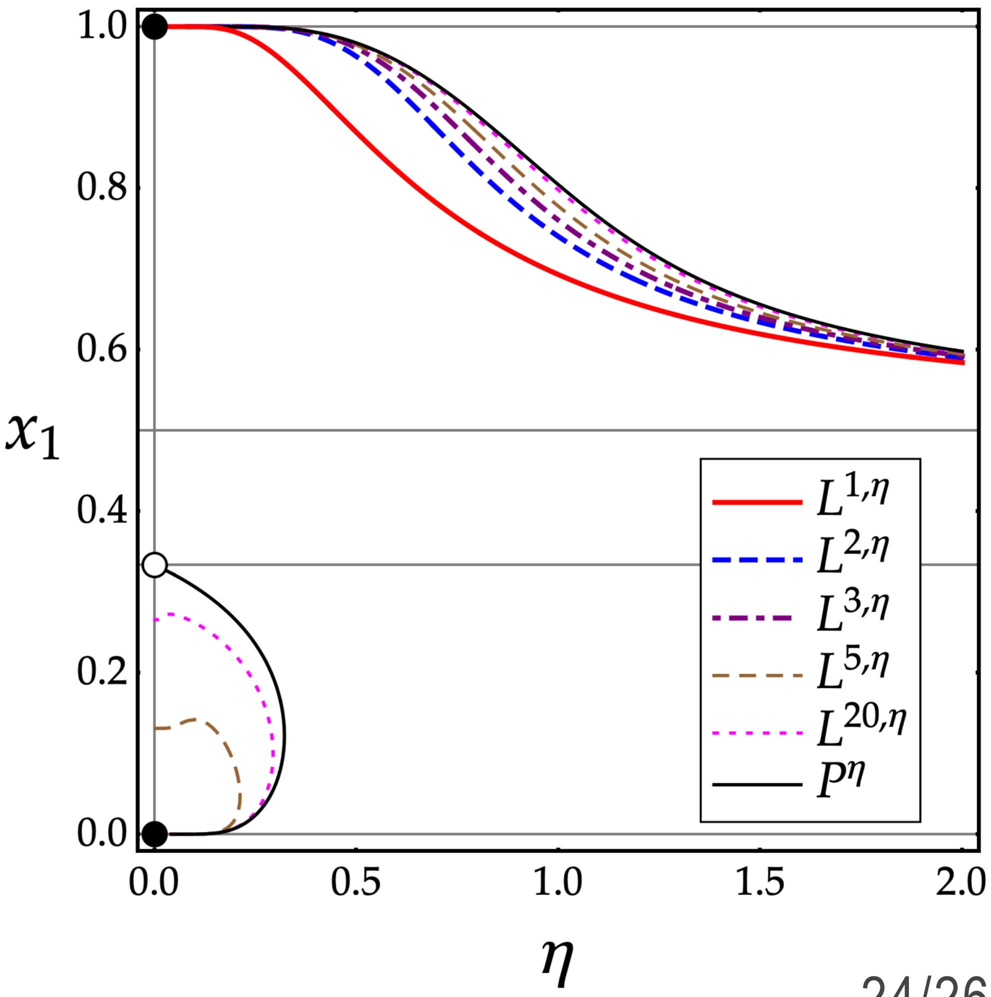
Example 1 (Cont'd): 2×2 Coordination Game

- Payoff estimation errors introduce **bias toward the suboptimal choice**.
- Comparative statics of the interior SLE $\widetilde{x}$:
  Let the interior NE be x^*_{int}:

$$\frac{\partial}{\partial \eta}\left|\widetilde{x} - x^*_{\text{int}}\right| > 0 \quad \text{for large } k$$

$$-\frac{\partial}{\partial k}\left|\widetilde{x} - x^*_{\text{int}}\right| < 0$$

- The region of attraction for the "upside" SLE enlarges in noisy environments (k small or η high)
  - cf. "Fast convergence" under medium η with partial observation (Kreindler–Young, 2013)



Summary

- Proposed and analyzed a new choice rule, the **sampling logit choie**, that combines two canonical noise sources.
- Thanks to the differentiability of logit, we obtained an intuitive interpretaion of choice/equilibrium distortion ("variance premium" / "virtual payoff").
- Still at an early stage. Many todos/extensions:
  - General characterization of equilibrium distortion/selection for some important class of games (e.g., stable games, potential games).
  - Application to concrete games with ≥ 3 actions (e.g., bilingual games)
  - Allowing random number of observations k: $L^{\eta}(x) \equiv \sum_{k=1}^{\infty} \lambda_k L^{k,\eta}(x)$
  - Endogenizing (k, η) via information/attention cost (rational inattention)
  - (Experimental validation)

Thank You for Your Attention! 🎉